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OCR
Oxford Cambridge and RSA

A Level Further Mathematics A
Y545 Additional Pure Mathematics
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 30 minutes

OCR supplied materials:
Printed Answer Booklet
Formulae A Level Further Mathematics A

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A
- Scientific or graphical calculator

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INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

1 A curve is given by $x = t^2 - 2\ln t$, $y = 4t$ for $t > 0$. When the arc of the curve between the points where $t = 1$ and $t = 4$ is rotated through 2π radians about the x-axis, a surface of revolution is formed with surface area A.

Given that $A = k\pi$, where k is an integer,

- write down an integral which gives A and
- find the value of k. **[4]**

need $\left(\frac{dx}{dt}\right)^2$ & $\left(\frac{dy}{dt}\right)^2$: $\frac{dx}{dt} = 2t - \frac{2}{t}$, $\frac{dy}{dt} = 4$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(2t - \frac{2}{t}\right)^2 + 4^2$$

$$A = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 2\pi \int_1^4 4t\sqrt{\left(2t - \frac{2}{t}\right)^2 + 16}\, dt$$

about x-axis

$$= 2\pi \int_1^4 4t\sqrt{\left(4t^2 - 8 + \frac{4}{t^2}\right) + 16}\, dt$$

$$= 2\pi \int_1^4 4t\sqrt{4t^2 + \frac{4}{t^2} + 8}\, dt$$

$$= 2\pi \int_1^4 4t\sqrt{\left(2t + \frac{2}{t}\right)^2}\, dt$$

$$= 2\pi \int_1^4 (8t^2 + 8)\, dt$$

$$= 2\pi \left[\frac{8}{3}t^3 + 8t\right]_1^4$$

$$= \underline{384\pi}$$

$\therefore k = 384$

2 Find the volume of tetrahedron OABC, where O is the origin, A = (2, 3, 1), B = (–4, 2, 5) and C = (1, 4, 4). [3]

$$V = \frac{1}{6}\underline{a}\cdot\underline{b}\times\underline{c}$$

$$\underline{b}\times\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 2 & 5 \\ 1 & 4 & 4 \end{vmatrix} = \underline{i}(8-20) - \underline{j}(-16-5) + \underline{k}(-16-2)$$

$$= (-12, 21, -18)$$

$$\Rightarrow V = \frac{1}{6}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -12 \\ 21 \\ -18 \end{pmatrix} = \underline{3.5}$$

3 Given $z = x\sin y + y\cos x$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + z = 0$. [5]

$$\left(\frac{\partial z}{\partial x}\right)_y = \sin y - y\cos x \quad \left(\frac{\partial z}{\partial y}\right)_x = x\cos y + \cos x$$

$$\frac{\partial^2 z}{\partial x^2} = -y\cos x \quad \frac{\partial^2 z}{\partial y^2} = -x\sin y$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + z = -y\cos x - x\sin y + x\sin y + y\cos x = 0$$

4 (i) Solve the recurrence relation $u_{n+2} = 4u_{n+1} - 4u_n$ for $n \geq 0$, given that $u_0 = 1$ and $u_1 = 1$. [4]

(ii) Show that each term of the sequence $\{u_n\}$ is an integer. [2]

i. characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow \lambda = 2 \ (\times 2)$

general solution: $u_n = (A + Bn) \times 2^n$

$u_0 = 1, u_1 = 1 \Rightarrow 1 = A, \ 1 = 2(A+B)$

$\therefore A = 1, \ B = -\frac{1}{2}$

$u_n = (1 - \frac{1}{2}n) \times 2^n$

ii. so $u_n = (2 - n) \times 2^{n-1}$ $(1 - \frac{1}{2}n) \times 2 \times 2^{n-1}$

the product is made of two integers

$\therefore u_n$ is an integer

5 **In this question you must show detailed reasoning.**

It is given that $I_n = \int_0^{\pi} \sin^n \theta \, d\theta$ for $n \geq 0$.

(i) Prove that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$. **[5]**

(ii) (a) Evaluate I_1. **[2]**

(b) Use the reduction formula to determine the exact value of $\int_0^{\pi} \cos^2\theta \, \sin^5\theta \, d\theta$. **[2]**

i. $I_n = \int_0^{\pi} \sin^n\theta \, d\theta \, d\theta = \int_0^{\pi} \sin\theta \cdot \sin^{n-1}\theta \, d\theta$

$= \left[-\cos\theta \cdot \sin^{n-1}\theta\right]_0^{\pi} - \int_0^{\pi} -\cos\theta \cdot (n-1)\sin^{n-2}\theta \cdot \cos\theta \, d\theta$

$u = \sin^{n-1}\theta$

$u' = (n-1)\sin^{n-2}\theta \cdot \cos\theta$

$v' = \sin\theta$

$v = -\cos\theta$

using

$\int f(\theta) = uv - \int u'v \, d\theta$

integration by parts

$= 0 + (n-1)\int_0^{\pi} \overbrace{(1-\sin^2\theta)}^{\cos^2\theta} \sin^{n-2}\theta \, d\theta$

$I_n = (n-1)(I_{n-2} - I_n)$

$\Rightarrow nI_n = (n-1)I_{n-2}$

$\Rightarrow I_n = \dfrac{n-1}{n} I_{n-2}$

ii. a) $I_1 = \int_0^{\pi} \sin\theta \, d\theta = \left[-\cos\theta\right]_0^{\pi} = 2$

b) $\cos^2\theta = 1 - \sin^2\theta \quad \therefore \quad I = \int_0^{\pi} (\sin^5\theta - \sin^7\theta) \, d\theta = I_5 - I_7$

$I_n = \dfrac{n-1}{n} I_{n-2} \Rightarrow I_7 = \dfrac{6}{7} I_5, \quad I_5 = \dfrac{4}{5} I_3 \quad I_3 = \dfrac{2}{3} I_1$

so $I = \left(1 - \dfrac{6}{7}\right) I_5 = \dfrac{1}{7} \times \dfrac{4}{5} I_3 = \dfrac{1}{7} \times \dfrac{4}{5} \times \dfrac{2}{3} I_1$

$= \dfrac{16}{105}$ using (a)

6 A surface S has equation $z = f(x, y)$, where $f(x, y) = 2x^2 - y^2 + 3xy + 17y$. It is given that S has a single stationary point, P.

(i) (a) Determine the coordinates of P. [5]

(b) Determine the nature of P. [3]

(ii) Find the equation of the tangent plane to S at the point $Q(1, 2, 38)$. [2]

i. a) $f_x = 4x + 3y$ $\qquad f_y = -2y + 3x + 17$

stationary point $\Rightarrow$ set to 0

$\underset{①}{4x+3y} = 0 = \underset{②}{-2y+3x+17}$

$2 \times ① + 3 \times ②$:

$8x + 9x + 51 = 0$

$\Rightarrow x = -3$

$\Rightarrow y = 4$

$2(-3)^2 - 4^2 + 3(-3)(4) + 17(4) = 34 = z$

b) $f_{xx} = 4 \quad f_{yy} = -2 \quad f_{xy} = f_{yx} = 3$

Hessian: $|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 3 & -2 \end{vmatrix} = -17$

$|H| < 0 \therefore P$ is a saddle point

ii. $z = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$

$= 38 + 10(x-1) + 16(y-2)$

$\Rightarrow 10x + 16y - z = 4$

7 In order to rescue them from extinction, a particular species of ground-nesting birds is introduced into a nature reserve. The number of breeding pairs of these birds in the nature reserve, t years after their introduction, is an integer denoted by N_t. The initial number of breeding pairs is given by N_0.

An initial discrete population model is proposed for N_t.

$$\text{Model I: } N_{t+1} = \tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)$$

(i) (a) For Model I, show that the steady state values of the number of breeding pairs are 0 and 150. **[3]**

(b) Show that $N_{t+1} - N_t < 150 - N_t$ when N_t lies between 0 and 150. **[3]**

(c) Hence find the long-term behaviour of the number of breeding pairs of this species of birds in the nature reserve predicted by Model I when $N_0 \in (0, 150)$. **[2]**

i. a) $N_{t+1} - N_t = \frac{1}{5} N_t - \frac{1}{750} N_t^2 = \frac{1}{750} N_t(150 - N_t)$

define steady state value as M

Steady state: no change between N_t & N_{t+1}

$\Rightarrow \frac{1}{750} M(150 - M) = 0$

$\therefore M = 0$ or 150

b) $N_{t+1} - N_t = \frac{1}{750} N_t (150 - N_t)$

$N_t \in (0, 150) \Rightarrow \frac{1}{750} N_t \in (0, \frac{1}{5})$

as $150 - N_t > 0$ $(0 < N_t < 150)$, $\frac{1}{750} N_t (150 - N_t)$ ranges between $0 \times$ & $\frac{1}{5} \times$ a +ve value

$\therefore N_{t+1} - N_t < 150 - N_t$ $\quad N_{t+1} - N_t =$ a fraction $\times (150 - N_t)$

c) $N_{t+1} - N_t > 0 \Rightarrow N_t$ increases & approaches 150 without oscillating

$= +\text{ve} \times +\text{ve}$

An alternative discrete population model is proposed for N_t.

$$\text{Model II: } N_{t+1} = \text{INT}\left(\tfrac{6}{5}N_t\left(1-\tfrac{1}{900}N_t\right)\right)$$

(ii) (a) Given that $N_0 = 8$, find the value of N_4 for each of the two models. **[2]**

(b) Which of the two models gives values for N_t with the more appropriate level of precision? **[1]**

ii. a) Model I: $N_1 = 9.51467$ $N_2 = 11.29689$ $N_3 = 13.38611$

$N_4 = 15.82442$

Model II: $N_1 = 9$ $N_2 = 10$ $N_3 = 11$

$N_4 = 13$

b) the number of pairs must be an integer, ∴ model II is the better suited model as it is more realistic

8 The set X consists of all 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where x and y are real numbers which are not **both** zero.

(i) **(a)** The matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ are both elements of X.

Show that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}=\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ for some real numbers p and q to be found in terms of a, b, c and d. **[2]**

(b) Prove by contradiction that p and q are not **both** zero. **[5]**

(ii) Prove that X, under matrix multiplication, forms a group G.
[You may use the result that matrix multiplication is associative.] **[4]**

(iii) Determine a subgroup of G of order 17. **[2]**

i. a) $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}=\begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$

$p=ac-bd$, $q=ad+bc$

b) set $p=q=0$: $\underset{①}{ac-bd=0}$, $\underset{②}{ad+bc=0}$

$d\times ①$: $acd-bd^2=0$

$c\times ②$: $acd+bc^2=0$

subtract: $b(c^2+d^2)=0$

c & d aren't both $0 \Rightarrow c^2+d^2\neq0$, $b=0$

$c\times ①$: $ac^2-bcd=0$

$d\times ②$: $ad^2+bcd=0$

add: $a(c^2+d^2)$

as before, c & d not both $0 \Rightarrow a=0$

Contradicts condition that a & b not both $0 \Rightarrow$ assumption is false

ii. part (i) gives closure condition, associativity given

identity given when $a=1, b=0$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -(-b) \\ -b & a \end{pmatrix}$$

inverse is $\in X$ $\because$ $a^2+b^2 \neq 0$ as a & b aren't both 0

iii. $a=\cos\left(\frac{2}{17}\pi\right)$, $b=\sin\left(\frac{2}{17}\pi\right)$ $\Rightarrow$ matrix generates

subgroup of rotations about O in

increments of $\frac{2}{17}\pi$

9 **(i) (a)** Prove that $p \equiv \pm 1 \pmod{6}$ for all primes $p > 3$. **[2]**

(b) Hence or otherwise prove that $p^2 - 1 \equiv 0 \pmod{24}$ for all primes $p > 3$. **[3]**

(ii) Given that p is an odd prime, determine the residue of 2^{p^2-1} modulo p. **[4]**

(iii) Let p and q be distinct primes greater than 3. Prove that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$. **[5]**

END OF QUESTION PAPER

i. a) if $p \equiv 0$ or $3 \pmod 6$ $\Rightarrow$ multiple of 3

if $p \equiv 0, 2,$ or $4 \pmod 6$ then it is even

i.e. not prime

$\therefore\ p \equiv \pm 1 \pmod 6$

b) $p = 6k \pm 1$

$$p^2 - 1 = (6k \pm 1)^2 - 1 = 36k^2 \pm 12k$$
$$= 12k(3k \pm 1)$$

if k is even $\Rightarrow 24 \mid 12k$

if k is odd $\Rightarrow 3k \pm 1$ is even so $24 \mid 12k(3k \pm 1)$

ii. $$2^{p^2-1} = 2^{(p-1)(p+1)}$$
$$= (2^{p-1})^{p+1}$$
$$= (1)^{p+1} \pmod p = 1$$

iii. $p^{q-1} \equiv 1 \pmod{q}$

$q^{p-1} \equiv 0 \pmod{q}$

$\Rightarrow p^{q-1} + q^{p-1} \equiv 1 \pmod{q}$

likewise, $p^{q-1} + q^{p-1} \equiv 1 \pmod{q}$

so, combine to give $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$

$\because \text{HCF}(p,q) = 1$